N_1 = mass flux in the case without chemical reaction
$\overline{N}, \overline{N}_1$ = temporal averages of N and N_1
t = time
u = x velocity component
u_0 = average velocity
v = y velocity component
w = propagation velocity of the roll-cells
U = average velocity at the interface with respect to x over a cell
x = distance along the interface
x_0 = length of a cell along the x direction
y = distance from the interface
z = $k(x - wt)$

Greek Letters

α_0 = quantity defined by Equation (5)
$\overline{\psi}$ = average with respect to time of the ratio between the mass flux when roll cells are acting and the mass flux resulting from the penetration theory
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
σ = surface tension
$\Delta\sigma$ = difference between the values of σ at the end and at the beginning of each cell
τ = shear stress at the interface
$\overline{\tau}$ = average of τ with respect to x

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The Periodic Viscous Sublayer Model and Heat Transfer to Drag Reducing Solutions

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In a recent paper (*11*), the Einstein-Li (*7*) concept of a periodic viscous sublayer at the bounding surfaces of a turbulent shear flow was reintroduced, and supporting experimental evidence was presented. The essential characteristics of the Einstein-Li model, which is a limiting case of a more rigorous model recently developed by Black (*1, 2*), are that, at a point on the wall, turbulent fluctuations suddenly penetrate to very near the surface, that these penetrations occur with a fairly well-defined regularity, and that between penetrations viscous processes dominate the sublayer flow. The virtue of such a treatment is that it allows prediction of heat and mass transfer to turbulently flowing fluids from the constitutive and state equations and physical properties (including any generation or rate constants), since only the well-known differential equations describing molecular transport need be solved. At the present stage of development of the model, the fluctuation period is obtained from friction factor data. Reference *10* may be consulted for a complete discussion and development of the model as well as critical evaluation of the assumptions involved.

THE SUBLAYER MODEL

Extensive data have been presented elsewhere (*10, 11*) in support of the periodic viscous sublayer model. The variable of principal interest is the mean period between turbulent penetrations T or, in dimensionless form, T^+, where $T^+ \equiv \sqrt{T/\nu}\, u_*$. (The fluid kinematic viscosity is designated ν and u_* is the friction velocity.) The result of the sublayer model calculation (*10, 11*) is that for Reynolds numbers N_{Re} greater than 10^4, the dimensionless mean period is essentially constant at 18. We may note immediately from this that the mean frequency of penetrations is proportional to u_*^2/ν.

It is of interest to compare this model prediction with the recent visual observations of Corino and Brodkey (*4*). Table 1 reproduces the data of Figure 8 of that paper, where average number of disturbances per second at a point on the wall have been converted to dimensionless mean periods. The agreement with the calculated value of $T^+ = 18$ is quite satisfactory.

TURBULENT FLOW OF AND HEAT TRANSFER TO DRAG REDUCING SOLUTIONS

Heat transfer to dilute polymer solutions which exhibit the Toms effect (*17*) has of late attracted a great deal of

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attention (*13, 15, 20* to *22*), and it has been shown that a considerable body of the polymer solution, heat transfer data is fit by a Chilton-Colburn type of analogy (*15*). In this note, application of the viscous sublayer model to dilute polymer solutions is described and indication is given of when a Chilton-Colburn analogy may and may not be expected to apply.

A valid description of the turbulent flow of slightly elastic solutions by use of the periodic sublayer model appears to be possible because it has been rather conclusively established that the effect of polymer addition on turbulent flow largely manifests itself as changes in the viscous sublayer, any modification to the core region being of secondary importance (*8, 12, 14, 18, 19*). In particular, it is known experimentally that the viscous sublayer is thickened relative to the pure solvent case even though the viscosity may be unchanged. From the point of view of the periodic model, this indicates that the viscous turbulent transition at the point on the wall is delayed, or, equivalently, that the period of growth is increased (*2, 10*).

The closure relationship for the model is the transition criterion for a sublayer patch. The attempts to deduce this information for Newtonian fluids met with little success (*10*), and the slightly elastic case appears even less promising. Some investigations of stability have been undertaken for elastic fluids (*3, 6, 9*), but the poorly understood coupling process between the sublayer and the turbulent core dominates the transition phenomenon (*1, 2, 9, 10*). It has been necessary to employ friction factor data in place of transition criteria in all calculations to date.

The complete exposition of the viscous sublayer model for a slightly elastic fluid has been undertaken (*10*). Inasmuch as the procedure is identical to that for Newtonian fluids (*10, 11*) and the mathematics is somewhat tedious, only an outline of important results is given here. The constitutive equation chosen to represent the fluid was a convected Maxwell equation, and the resultant differential equations were solved by use of the Laplace transform. The bounding velocity at the outside of the sublayer u_B, the sublayer thickness δ_M, and the period T are found to be related by

$$u_B = \frac{f<u>^2}{4}\left(\frac{\pi T}{\nu}\right)^{1/2}\left(1+\frac{t_0}{T}+\ldots\right) \qquad (1)$$

and

$$\delta_M = 3.64\,(\nu T)^{1/2}\left(1-3\frac{t_0}{T}+\ldots\right) \qquad (2)$$

where the functions appearing in the development have been suitably expanded in t_0/T (*10*). These are the analogues of Equations (5) and (6) of reference *11* and of course revert to them when $t_0 = 0$.

In principle, the development could proceed as for the Newtonian case if the core is presumed completely unaffected. The complexity of the ensuing equations and the fact that no relationship equivalent to the Prandtl friction factor relationship exists for drag reducing solutions precludes complete solution of the problem. However, it can be shown (*10*) when t_0 is substantially less than T that if a drag reduction results, then the thickness and period increase.

It is of interest to compare dimensionless periods for the solution and solvent in a given tube at the same flow rate. As is shown in reference *11*, Figure 2, u_B is well into the zone where velocity is not very dependent on distance from the wall. That is, to a first approximation, if the sublayer is not thickened too much, u_B is the same for both solvent and solution. Further suppose that physical properties are unchanged (that is, the solution is very dilute) and that $t_0 << T$. Then, from (1)

$$\frac{T_v}{(T_s)^{1/2}} = \frac{f_s}{f_v} \qquad (3)$$

where the subscript v signifies the viscoelastic case and s the pure solvent. It may be noted that this development is equivalent to assuming at the outset that $\delta_M << R$; that is, that the turbulent flow is fully developed and that almost all of the contribution to $<u>$ comes from the core (*10*). The assumption is a restriction on the following discussion.

Sublayer periods have been measured and reported elsewhere (*11*) for turbulently flowing pure Tetralin and for a 0.02 percent by weight solution of polyisobutylene in Tetralin (*10, 11*). (For the solution: $t_0 \sim 10^{-4}$ sec., $T \sim 10^{-2}$ sec., $\delta_M/R \sim 0.05$, ν unchanged.) Table 2 shows that Equation (3) applies as expected.

The next, and most important, consideration is the dependence of the heat transfer coefficient on the sublayer period for a constant u_B. For the hypothetical case where the turbulent penetrations extend completely to the wall, it is easy to show that (*10*)

$$N_{St} \propto T^{-1/2} \qquad (4)$$

for both the cases of constant wall temperature and constant flux. (N_{St} is the local Stanton number). In fact, the minimum thickness is not zero, and an erroneous square-root dependence of the Stanton number on the Prandtl number arises if turbulence penetration to the wall is assumed. It has been shown (*10, 11*) that correct Prandtl number dependence results if a minimum dimensionless sublayer thickness of 1.5 is assumed. This value of the minimum thickness is confirmed experimentally. Wall temperature fluctuations and energy accumulation in the fluid were treated by obtaining numerical solutions to the steady periodic problem (*10, 11*), and the numerically calculated heat transfer coefficients for Newtonian fluids are in excellent agreement with Deissler's correlation (*5*).

TABLE 1. DIMENSIONLESS SUBLAYER PERIOD VS. REYNOLDS NUMBER FROM DATA OF REFERENCE *4*

N_{Re}	T^+
$2\,(10)^4$	15
$3\,(10)^4$	15
$4\,(10)^4$	14
$4.5\,(10)^4$	17
$5.5\,(10)^4$	17

TABLE 2. COMPARISON OF $(T_v/T_s)^{1/2}$ AND (f_s/f_v) FROM REFERENCE (*10*)

N_{Re}	$(T_v/T_s)^{1/2}$	(f_s/f_v)
58,700	1.3	1.2
64,400	1.3	1.3
74,400	1.5	1.3

It is also straightforward to write down analytical solutions for the cases of constant wall temperature or constant flux if the initial temperature distribution in the region between the wall and the minimum thickness is assumed linear. The Prandtl number dependence predicted by these two equations brackets the correct (numerically calculated ≡ experimental) dependence, and for not too large values of the parameter

$$\frac{\delta_1{}^+ \sqrt{N_{Pr}}}{2T^+}$$

Equation (4) once again obtains. For large values of the parameter (Prandtl numbers greater than about 10) no such simple behavior is evidenced. It can be noted that the minimum sublayer thickness at a given flow is approximately unchanged as evidenced by the fact that wall temperature fluctuation data for Tetralin and polyisobutylene-Tetralin solution (see reference *11*, Figure 7) show that within the experimental uncertainty the relative temperature fluctuation magnitude is unchanged.

When Equations (3) and (4) are combined, it is found that

$$\frac{N_{St,v}}{N_{St,s}} = \frac{f_v}{f_s} \qquad (5)$$

Thus, when the solvent follows a Chilton-Colburn analogy

$$N_{St,s} = \frac{f_s}{2} N_{Pr}{}^{-0.6} \qquad (6)$$

Then, immediately

$$N_{St,v} = \frac{f_v}{2} N_{Pr}{}^{-0.6} \qquad (7)$$

that is, solution and solvent manifest the same momentum transfer, heat transfer analogy. In particular, Chilton-Colburn behavior should be, and is, (*15*), found for dilute water solutions at large Reynolds numbers.

The assumptions required to obtain Equation (5) are worth restating:

1. The solution must be so dilute that physical properties such as viscosity and thermal conductivity are unaltered.
2. The polymer properties and flow conditions must be such that the solution relaxation time is much less than the sublayer growth period.
3. The Reynolds number must be large enough so that the sublayer thickness is much less than the tube radius.
4. The Prandtl number should not be large.

Thus, while there are certainly a large number of situations where the solution follows the same momentum heat transfer analogy as the solvent, this result cannot be presumed in general. The assumptions listed above indicate the likely, but not exclusive, circumstances for which heat transfer to viscoelastic solutions can be described by simpler momentum, heat transfer analogies. Unfortunately, it is not possible to test these predictions further by use of available data, since the fluids have seldom been completely characterized in a form suitable for this analysis, and a wide enough range of fluid properties has not yet been considered.

NOTATION

C = heat capacity
D = diameter of tube
f = friction factor
h = local heat transfer coefficient based on bulk fluid temperature
k = thermal conductivity
N_{Nu} = local Nusselt number based on bulk fluid temperature, hD/k
N_{Pr} = fluid Prandtl number, ν/α
N_{Re} = Reynolds number based on bulk velocity, $D{<}u_z{>}/\nu$
N_{St} = local Stanton number, $N_{Nu}/N_{Re}N_{Pr}$
q = heat flux; $\overline{q}_w$, time averaged heat flux at the wall
R = tube radius; $R^+ = Ru_*/\nu$
t = time; the dimensionless time from start of the growth period, $t^+ = u_*\sqrt{t/\nu}$; the relaxation time, t_0
T = sublayer growth period
u = velocity; $<u>$, bulk velocity; u_z, velocity component in z direction; u_B, velocity at edge of growing sublayer; $u_* = \sqrt{\tau_w/\rho}$; $u^+ = u/u_*$
x, y, z = rectangular Cartesian coordinates relative to a point on the wall in the transverse, normal and axial directions of a tube respectively; $y^+ = yu_*/\nu$

Greek Letters

α = thermal diffusivity; $k/\rho C$
δ = thickness of growing sublayer; $\delta^+ = \delta u_*/\nu$; δ_M, the maximum thickness of sublayer growth; δ_1, the minimum sublayer decay thickness
θ = temperature; θ_B, constant temperature at the edge of the growing thermal sublayer; $<\theta>$, bulk fluid temperature; $\overline{\theta}_w$, time averaged wall temperature
μ = fluid viscosity
ν = fluid kinematic viscosity, μ/ρ
ρ = density
τ = shear stress; τ_w, time average wall shear stress

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